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MONTE CARLO ANALYSIS OF NONLINEAR VIBRATION OF RECTANGULAR PLATES WITH RANDOM GEOMETRIC IMPERFECTIONS

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Abstract—The effect of random initial geometric imperfections on the vibration behavior of rectangular plates is investigated in this paper using a statistical method. The random initial geometric imperfections of plates are described by *Gaussian* random fields and simulated numerically using *Elishakoff*'s method. *Lindstedt-Poincaré*'s perturbation technique is employed to solve *Duffing*'s Equation with an additional quadratic spring term derived in the vibration analysis of imperfect rectangular plates. A *Monte Carlo* analysis for simply supported plates is carried out in detail to illustrate the proposed approach. It is shown that the effect of random geometric imperfections on the vibration behavior of the plates can be described quantitatively in terms of the frequency reliability function and the hardening type probability.

1. INTRODUCTION

It is well known that the initial geometric imperfection has a profound effect on the buckling behavior of shell and plate structures; however, the effect of imperfection on the vibrational behavior has received relatively little attention. This may be caused by the fact that a considerable amount of work on nonlinear vibrations of perfect structures (Reissner, 1955; Chu and Herrmann, 1956; Chu, 1961; Yamaki, 1961; Prathap and Veradan, 1978) has shown that the large amplitude vibrations of the perfect structures were always of hardening type for various boundary conditions and shapes of the structures.

Recently, the effect of initial geometric imperfection has been investigated by several authors (Rosen and Singer, 1974; Singer and Prucz, 1982; Hui, 1983, 1984a, b; Hui and Leissa, 1983a, b). A series of works have been published by Hui and Leissa (1983a, b) and Hui (1983, 1984a, b). One term mode was used in their papers for both the vibration and imperfection modes and it was found that the presence of geometric imperfection may significantly raise the free vibration frequency. More interesting, contrary to the well-established and widely accepted theory that the nonlinear vibration of flat plates is of the hardening type, it was shown that the presence of unavoidable geometric imperfection amplitudes of only half the plate thickness may change the nonlinear hard-spring character of the plate to one with a soft-spring behavior.

This paper is based on the work reported in Wang (1986). Two further questions about the description of imperfections and the corresponding analysis are investigated in the paper. The first is about the form of imperfection. Clearly, the initial imperfection is independent of the vibration mode and only depends on the manufacturing process, environment, etc. The second is that in practice it is very difficult to describe imperfection exactly. In most cases, it is imperative to take the imperfection as a random variable or field. Consequently, the uncertainties of imperfection must be taken into account in studying its effects on the vibrational behavior.

The paper suggests a statistical method using the *Monte Carlo* technique which can be used to solve the two questions in a unified way. The method by Elishakoff and Arbocz (1982) for the buckling analysis of shell structures is used, and is demonstrated in detail through the vibration simulation analysis of simply supported rectangular plates with random geometric imperfections. The *Monte Carlo* method is used to "create" a large number of imperfection plate samples, and the distribution function of the vibration fre-

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quency and the probability by which a plate behaves like a hard-spring or a soft-spring, the two most important characteristics of vibration behavior, are calculated in the simulation.

2. MOTION EQUATIONS OF IMPERFECT PLATES

The dynamic analogue of *von Karman* equilibrium and compatibility differential equations (also known as *Marguerre* equations) in terms of the normal displacement $\bar{w}$ and the *Airy* stress function F for moderately large amplitude vibrations of plates with geometric imperfection $\bar{w}_0$ can be given by (Chia, 1980)

$$D\bar{\nabla}^4\bar{w}+\rho\frac{\partial^2\bar{w}}{\partial\bar{t}^2} = \bar{q}(\bar{x},\bar{y},\bar{t})+h[F_{,\bar{x}\bar{x}}(\bar{w}_{0,\bar{y}\bar{y}}+\bar{w}_{,\bar{y}\bar{y}})+F_{,\bar{y}\bar{y}}(\bar{w}_{0,\bar{x}\bar{x}}+\bar{w}_{,\bar{x}\bar{x}})-2F_{,\bar{x}\bar{y}}(\bar{w}_{0,\bar{x}\bar{y}}+\bar{w}_{,\bar{x}\bar{y}})] \quad (1)$$

$$\bar{\nabla}^4 F = E[(\bar{w}_{,\bar{x}\bar{y}})^2-\bar{w}_{,\bar{x}\bar{x}}\bar{w}_{,\bar{y}\bar{y}}-\bar{w}_{0,\bar{x}\bar{x}}\bar{w}_{,\bar{y}\bar{y}}-\bar{w}_{0,\bar{y}\bar{y}}\bar{w}_{,\bar{x}\bar{x}}+2\bar{w}_{0,\bar{x}\bar{y}}\bar{w}_{,\bar{x}\bar{y}}] \quad (2)$$

where

$$\bar{\nabla}^2 = \frac{\partial^2}{\partial\bar{x}^2}+\frac{\partial^2}{\partial\bar{y}^2},$$

D is the flexural rigidity, E is *Young*'s modulus, h is the plate thickness, ρ is the plate mass per unit area, $\bar{x}$, $\bar{y}$ are the two in-plane coordinates and $\bar{t}$ is the time.

We introduce now the nondimensional quantities w, w_0, f, q, x, y and t, which are defined by

$$\begin{aligned}(w,w_0) &= (\bar{w},\bar{w}_0)/h, \quad f = F/Eh^2\\ (x,y) &= (\bar{x},\bar{y})/b, \quad t=\omega_0\bar{t}\\ q(x,y,t) &= \bar{q}(\bar{x},\bar{y},\bar{t})/q_0\end{aligned} \quad (3)$$

and

$$c=\sqrt{3(1-\nu^2)}, \quad \omega_0^2=\frac{4\pi^4D}{\rho b^4}=\frac{\pi^4Eh^3}{\rho c^2b^4},$$
$$\alpha=\frac{a}{b}, \quad q_0=\frac{4\pi^4Dh}{b^4}=\frac{\pi^4Eh^4}{c^2b^4}$$

where ν is Poisson's ratio and a and b are the plate widths along the x- and y-directions, respectively. The governing nonlinear differential equations (1), (2) can now be written in the nondimensional form

$$\nabla^4w+4\pi^4\frac{\partial^2w}{\partial t^2} = 4\pi^4q+4c^2[F_{,xx}(w_{0,yy}+w_{,yy})+F_{,yy}(w_{0,xx}+w_{,xx})-2F_{,xy}(w_{0,xy}+w_{,xy})] \quad (4)$$

$$\nabla^4F = (w_{,xy})^2-w_{,xx}w_{,yy}-w_{0,xx}w_{,yy}-w_{0,yy}w_{,xx}+2w_{0,xy}w_{,xy}, \quad (5)$$
$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

The boundary conditions are taken as simply supported, the in-plane displacements normal to the edges are constant and there is no in-plane shear along all edges, that is,

$$\begin{aligned}&x=0 \text{ or } \alpha \quad w=0, \quad w_{,xx}=0, \quad f_{,xx}=0, \quad f_{,xy}=0;\\ &y=0 \text{ or } 1 \quad w=0, \quad w_{,yy}=0, \quad f_{,yy}=0, \quad f_{,xy}=0.\end{aligned} \quad (6)$$

For a rectangular plate, the fundamental vibrational mode corresponds to the half sine waves on both the x- and y-directions. Some interesting results have been obtained by Hui (1984a) by assuming that the shape of the geometrical imperfection is the same as the fundamental mode. Obviously, this is not the general case. In fact a realistic geometrical imperfection cannot be described deterministically in most cases. In order to get a complete picture of the dynamic behavior of structures a statistical analysis of the imperfection must be used.

3. REPRESENTATION OF RANDOM GEOMETRIC IMPERFECTIONS

In actual structures the initial geometric imperfections are unavoidable and cannot be described deterministically in general. In fact, the magnitude or type of geometric imperfection of practical structures are never known exactly and, in a mass production situation, these quantities will generally be subject to random variations. To obtain a clear and complete picture of the behavior of such imperfect structures it becomes imperative to take account of uncertainties that enter into any real application. The purely analytical approach is to use a random field to describe the initial imperfection. This approach is rigorous in its concept, but has disadvantages resulting from the mathematical complexities. A method of digital simulation of *Gaussian* random fields has been developed by Elishakoff (1979). In this method, the original random field problem is reduced to one of the simulation of normal vectors. This method is very efficient in view of the difficulties in the purely analytical approach, and of the recent advance of high-speed digital computers. Here, we extended the method to describe the initial geometric imperfection in the plate structures.

Consider a plate which occupies the region Ω. Let $w_0(\mathbf{x})$ be a *Gaussian* random field on Ω, and $\mathbf{x} = (x_1, x_2)$ be a coordinate vector. Suppose $w_i(\mathbf{x})$ $(i = 1, \ldots, \infty)$ is a complete set of orthogonal functions on Ω which satisfy certain boundary conditions. Thus, the random field $w_0(\mathbf{x})$ can be represented by the following series:

$$w_0(\mathbf{x}) = \sum_{i=1}^{\infty} A_i w_i(\mathbf{x}) \tag{7}$$

where A_i, $i = 1, \ldots, \infty$, are normal random variables and they are correlated with each other. In practical computations, the series of eqn (7) is usually truncated to some finite number N, so that the equation is replaced in what follows by

$$w_0(\mathbf{x}) = \sum_{i=1}^{N} A_i w_i(\mathbf{x}). \tag{8}$$

The mean value function of $w_0(\mathbf{x})$ then becomes

$$E[w_0(\mathbf{x})] = \sum_{i=1}^{N} E(A_i) w_i(\mathbf{x}).$$

Thus the mean values of the A_is are readily found as

$$E(A_i) = (E[w_0(\mathbf{x})], w_i(\mathbf{x}))/(w_i(\mathbf{x}), w_i(\mathbf{x})), \quad i = 1, \ldots, N. \tag{9}$$

where

$$(\phi, \psi) = \int_{\Omega} \phi\psi \, \mathrm{d}\Omega$$

is the *inner product* of ϕ and ψ.

The variance–covariance matrix of A_is can be determined by the *auto-correlation* function of $w_0(\mathbf{x})$,

$$R(\mathbf{x}_1, \mathbf{x}_2) = E\{(w_0(\mathbf{x}_1) - E[w_0(\mathbf{x}_1)])(w_0(\mathbf{x}_2) - E[w_0(\mathbf{x}_2)])\} = \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij} w_i(\mathbf{x}_1) w_j(\mathbf{x}_2), \quad (10)$$

where

$$\sigma_{ij} = E\{(A_i - E[A_i])(A_j - E[A_j])\}. \quad (11)$$

Using the orthogonality of $w_i(\mathbf{x})$s, we find

$$\sigma_{ij} = \frac{1}{(w_i, w_i)(w_j, w_j)} \int_\Omega \int_\Omega R(\mathbf{x}_1, \mathbf{x}_2) w_i(\mathbf{x}_1) w_j(\mathbf{x}_2) \, d\Omega^2, \quad i, j = 1, \ldots, N. \quad (12)$$

The problem is now reduced to simulation of the random vector $\{A\}^T = \{A_1, A_2, \ldots, A_N\}$ with mean value (9) and variance–covariance matrix $[\Sigma] = (\sigma_{ij})_{N \times N}$ defined by eqn (12). Because of its positive definiteness, this matrix has a unique *Cholesky* decomposition; that is, there exists a lower triangular matrix $[C]$ with positive diagonal elements which satisfies

$$[\Sigma] = [C][C]^T. \quad (13)$$

Then, according to the theory of probability, the random vector $\{A\}$ can be represented as

$$\{A\} = [C]\{B\} + \{\bar{A}\} \quad (14)$$

where $\{\bar{A}\}$ is the mean value vector of $\{A\}$, and $\{B\}^T = \{B_1, B_2, \ldots, B_N\}$ is the standard normal random vector with distribution $N(\{0\}, [I_N])$. Therefore the problem can be further reduced to one of the simulation of a set of independent standard normal variables.

Since eqn (8) is an approximation of the random field $w_0(\mathbf{x})$, by constructing the mean function and the auto-correlation function of $w_0(x)$ from the collected data of the real manufacturing process, the mean value and the variance–covariance matrix of A_is can be calculated directly through eqns (9) and (12). This idea, even though it may be expensive, is very useful sometimes.

4. LARGE AMPLITUDE VIBRATION ANALYSIS OF IMPERFECT PLATES

We now apply the above-mentioned results to the present problem. The initial random geometric imperfection can be described approximately as

$$w_0(x, y) = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} A_{ij} \sin I\pi x \sin j\pi y, \quad I = i/\alpha. \quad (15)$$

A_{ij} in the equation are normal random variables.

Let the fundamental vibration mode and the pressure distribution be given in form

$$\{w(x, y, t), q(x, y, t)\} = \{w(t), q_1 \cos(\omega t/\omega_0)\} \sin M\pi x \sin n\pi y$$

where ω is the vibration frequency, $M = m/\alpha$, and $w(t)$ is the time-depended amplitude of $w(x, y, t)$. The specification of this type of pressure distribution will not affect the problem, since the present paper deals primarily with free vibrations of the plate.

Substituting $w(x, y, t)$ and $w_0(x, y)$ into the nonlinear compatibility eqn (5), it follows that

$$\nabla^4 f = \frac{\pi}{2} M^2 n^2 (\cos 2M\pi x + \cos 2n\pi y) + \pi^4 w(t) \left[2MnC(m,n) \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} A_{ij} IjC(i,j) - S(m,n) \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} A_{ij}(I^2 n^2 + j^2 M^2) S(i,j) \right].$$

Solving this equation with the boundary conditions (6), we find the stress function f to be

$$f(x,y,t) = \tfrac{1}{32} w^2(t) \left[\left(\frac{n}{M}\right)^2 \cos 2M\pi x + \left(\frac{M}{n}\right)^2 \cos 2n\pi y \right] + \tfrac{1}{4} w(t) \left\{ \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} A_{ij} [H^1_{ijmn} C(i+m, j+n) + H^2_{ijmn} C(i-m, j-n) + H^3_{ijmn} C(i+m, j-n) + H^4_{ijmn} C(i-m, j+n)] \right\} \quad (16)$$

where the notations $C(i,j)$, $S(i,j)$ and H^k_{ijmn} are

$$C(i,j) = \cos I\pi x \cos j\pi y, \qquad S(i,j) = \sin I\pi x \sin j\pi y,$$

$$H^1_{ijmn} = -\frac{\Delta^1_{ijmn}}{\Delta(i+m, j+n)}, \qquad H^2_{ijmn} = -\frac{\Delta^1_{ijmn}}{\Delta(i-m, j-n)},$$

$$H^3_{ijmn} = \frac{\Delta^2_{ijmn}}{\Delta(i+m, j-n)}, \qquad H^4_{ijmn} = \frac{\Delta^2_{ijmn}}{\Delta(i-m, j+n)},$$

$$\Delta^k_{ijmn} = [In + (-1)^k jM]^2, \; k = 1,2; \quad \Delta(i,j) = (I^2 + j^2)^2. \quad (17)$$

Substituting $w(x,y,t)$, $w_0(x,y)$, $q(x,y,t)$ and $f(x,y,t)$ into the nonlinear equation of motion (4) and applying the *Galerkin* procedure, i.e. multiplying both sides by $S(m,m)$ and then integrating over the plate, we obtain the following second-order nonlinear ordinary differential equation in time for the amplitude $w(t)$:

$$\frac{d^2 w}{dt^2} + \frac{1}{4} \left[(M^2 + n^2)^2 - \frac{c^2}{4} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} A_{ij} F_{ijmn} \right] w + \frac{3c^2}{16} [M^4 (A_{mn} - A_{m3n}) + n^4 (A_{mn} - A_{3mn})] w^2 + \frac{c^2}{16} (M^4 + n^4) w^3 = q_1 \cos(\omega t/\omega_0) \quad (18)$$

where

$$F_{ijmn} = \sum_{k=1}^{4} H^k_{ijmn} F^k_{ijmn} \quad (19)$$

$$F^1_{ijmn} = \Delta^1_{ijmn}[A_{ij} + A_{(i+2m)(j+2n)}] - \Delta^5_{ijmn}[A_{i(j+2n)} + A_{(i+2m)j}]$$

$$F^2_{ijmn} = \Delta^1_{ijmn}[A_{ij} + A_{(2m-i)(2n-j)} + A_{(i-2m)(j-2n)} - A_{(2m-i)(j-2n)} - A_{(i-2m)(2n-j)}] + \Delta^6_{ijmn}[A_{i(2n-j)} + A_{(2m-i)j} - A_{i(j-2n)} - A_{(i-2m)j}]$$

$$F^3_{ijmn} = \Delta^2_{ijmn}[A_{(i+2m)(2n-j)} - A_{(i+2m)(j-2n)} - A_{ij}] + \Delta^7_{ijmn}[A_{(i+2m)j} + A_{i(j-2n)} - A_{i(2n-j)}]$$

$$F^4_{ijmn} = \Delta^2_{ijmn}[A_{(2m-i)(2n+j)} - A_{(i-2m)(j+2n)} - A_{ij}] + \Delta^8_{ijmn}[A_{i(j+2n)} + A_{(i-2m)j} - A_{(2m-i)j}]$$

$$\Delta^{4+k}_{ijmn} = [In - (-1)^k 2Mn + jM]^2, \quad \Delta^{6+k}_{ijmn} = [In - (-1)^k 2Mn - jM]^2, \quad k = 1,2.$$

$$A_{ij} = 0, \quad \text{if } i \leqslant 0, \quad \text{or } i > N_1, \quad \text{or } j \leqslant 0, \quad \text{or } j > N_2. \quad (20)$$

Letting the random parameters δ, ε, ξ be

$$\delta = \frac{1}{4}\left[(M^2+n^2)^2 - \frac{c^2}{4}\sum_{i,j=1}^{N_1N_2} A_{ij}F_{ijmn}\right], \quad \varepsilon = \frac{c^2}{16}(M^4+n^4)\,\delta$$

$$\xi = 3\left(A_{mn} - \frac{M^4A_{m3n}+n^4A_{3mn}}{M^4+n^4}\right), \tag{21}$$

eqn (18) becomes

$$\frac{\mathrm{d}^2w}{\mathrm{d}t^2} + \delta w + \varepsilon\delta(\xi w^2 + w^3) = q_1 \cos(\omega t/\omega_0) \tag{22}$$

which is the well-known *Duffing*'s Equation with an additional quadratic spring term. As in Nayfeh and Mook (1979), considering the case $q_1 = 0$, taking ε as a small parameter and using *Lindstedt-Poincaré*'s perturbation method, we get that the ratio of the nonlinear to linear free vibration frequencies $\Omega/\Omega_0(\Omega_0^2 = \delta)$ is related to the vibration amplitude A in the form

$$\Omega/\Omega_0 = 1 + \eta A^2 - 15\varepsilon^2A^4/256, \tag{23}$$

where

$$\eta = 3\varepsilon/8 - 5\xi^2\varepsilon^2/12. \tag{24}$$

The result here is same in the form as that obtained by Hui (1984a); however, from the equations in (21), the parameters ε, ξ, therefore η are now random variables. Note that the random parameter η is a characteristic parameter for the dynamic behavior of the plates, since, at least in the case of sufficiently small values of the vibration amplitude A, the nonlinear hard-spring or soft-spring characters can be indicated by positive or negative values of the random parameter η, respectively.

5. MONTE CARLO SIMULATION AND NUMERICAL RESULTS

As shown by formula (14), having S realizations of random vector $\{B\}$, we obtain the same number of realizations of $\{A\}$, i.e. the realizations of plate samples. This simulation technique is applicable for homogeneous as well as for nonhomogeneous random processes with the given mean and autocovariance functions.

For a numerical example, assume the random initial imperfection is fully separable, and the autocovariance function is of exponential–cosine type in both the x- and y-direction, i.e. the dimensionless autocovariance function is

$$R(\mathrm{x}_1,\mathrm{x}_2) = \Delta^2 \exp(-A_1|x_1-x_2| - A_2|y_1-y_2|)\cos B_1(x_1-x_2)\cos B_2(y_1-y_2). \tag{25}$$

The mean function is taken as

$$E[w_0] = \mu \sin K\pi x \sin l\pi y; \tag{26}$$

this form emphasizes the effect of imperfection mode (k, l).

Substituting the two functions into eqns (9), (12), we obtain expressions for $\bar{A}_{ij}$ and σ_{ijkl} as follows:

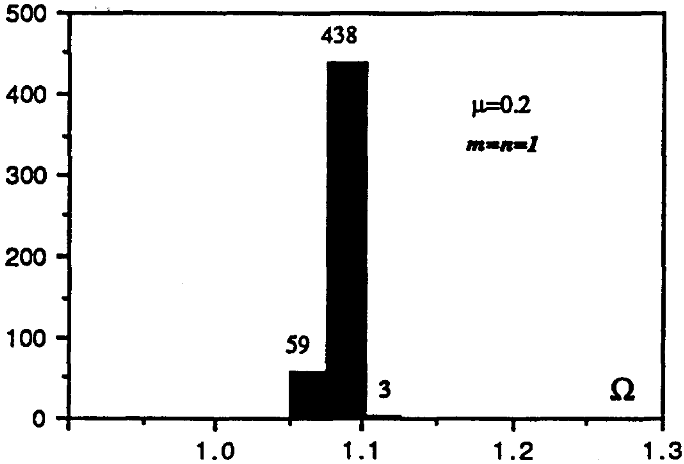

Fig. 1. Histogram of frequency.

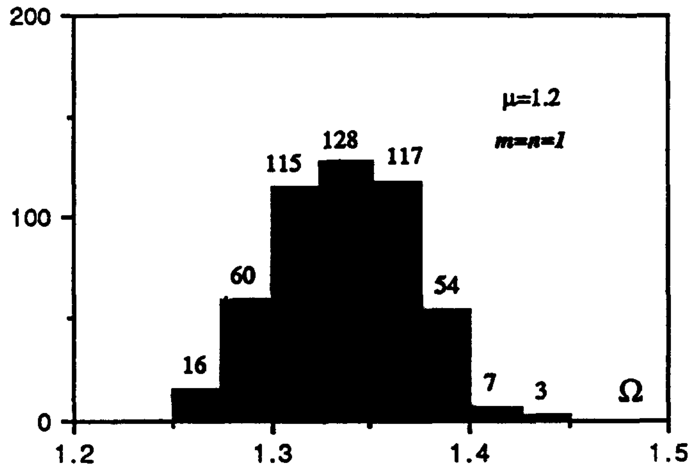

Fig. 2. Histogram of frequency.

$$\bar{A}_{ij} = \mu\delta_{ik}\,\delta_{jl}, \quad \sigma_{ijkl} = 4\Delta^2 I_{ik}(A_1, B_1) I_{jl}(A_2, B_2) \tag{27}$$

$$I_{ij}(A,B) = \begin{cases} A\dfrac{\Delta_{3i}+2i\pi B}{\Delta_i} + \dfrac{1}{\Delta_i}\left[\left(\dfrac{\Delta_{1i}\Delta_{4i}^2-\Delta_{2i}\Delta_{3i}^2}{\Delta_i}+2B\right)AG_{1i}\right. \\ \left. +\left(\dfrac{\Delta_{5i}\Delta_{4i}^2-\Delta_{6i}\Delta_{3i}^2}{2\Delta_i}+\Delta_{1i}\Delta_{2i}-A^2\right)G_{2i}\right] & i=j, \\ 2\dfrac{1}{i^2-j^2}\left(\dfrac{i^2\rho_i}{\Delta_i}-\dfrac{j^2\rho_j}{\Delta_j}\right) & i\neq j \end{cases}$$

$$\Delta_{ki} = B-(-1)^k i\pi, \quad \Delta_{(2+k)i} = A^2+\Delta_{ki}^2, \quad \Delta_{(4+k)i} = A^2-\Delta_{ki}^2, \quad \Delta_i = \Delta_{3i}\Delta_{4i}, \quad k=1,2.$$

$$\rho_i = 2ABG_{1i}-(A^2-\Delta_{1i}\Delta_{2i})G_{2i}, \quad G_{1i} = (-1)^i e^{-A}\sin B, \quad G_{2i} = (-1)^i e^{-A}\cos B - 1.$$

Equation (27) is analogous to Elishakoff's formula for the simply supported beam in Elishakoff [1983, eqns (10.84) or (11.36)].

The calculations are carried out for various types of mean imperfection and vibration mode. The *Monte Carlo* Method (Mihram, 1972) is applied to generate 500 realizations of

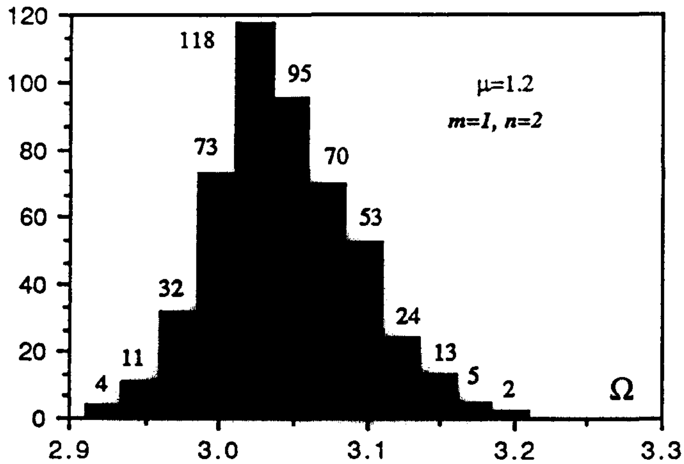

Fig. 3. Histogram of frequency.

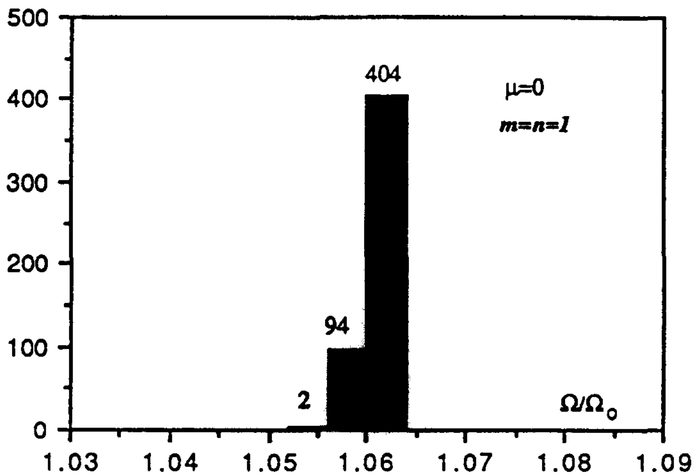

Fig. 4. Histogram of frequency ratio.

the random vector $\{B\}$; thus the same number of plate samples are "created". The parameters of the simulated plates are fixed at $\nu = 0.3$, $A_1 = A_2 = \pi/2$, $B_1 = B_2 = \pi$, $\Delta^2 = 0.005$. For each plate sample a deterministic vibration analysis is performed. Some of the simulation results are shown in Figs 1–8 ($A^2 = 0.5$, $k = l = 1$, $N_1 = N_2 = 8$ for all figures).

The numerical results show that the degree of divergence of frequency (i.e. the deviation from the normal concentration of frequency) increases as the fundamental vibration mode (m, n) or the mean imperfection amplitude μ increase (Figs 1–3). More interesting, for the given vibration mode (m, n) and mean imperfection mode (k, l), there exists a specific value μ_c of μ; the degree of divergence of frequency ratio Ω/Ω_0 is significantly high near μ_c and significantly low far away from μ_c (Figs 4–6). In the case $(k, l) = (m, n)$, considering only the vibration mode $S(m, n)$ term in the geometric imperfection series (15), we can find approximately this specific mean imperfection amplitude value to be

$$\mu_c \approx \sqrt{E[A_{mn}^2 \,|\, \eta(A_{mn}^2) = 0]} \approx \pm\frac{1}{c}\left(\frac{1}{2} + \frac{\zeta}{1+\zeta^2}\right)^{1/2}, \quad \zeta = \left(\frac{n}{M}\right)^2. \tag{28}$$

The simulation has shown that η is small when $\mu = \mu_c$, which means the vibrational behavior is very sensitive near the region $\mu = \mu_c$.

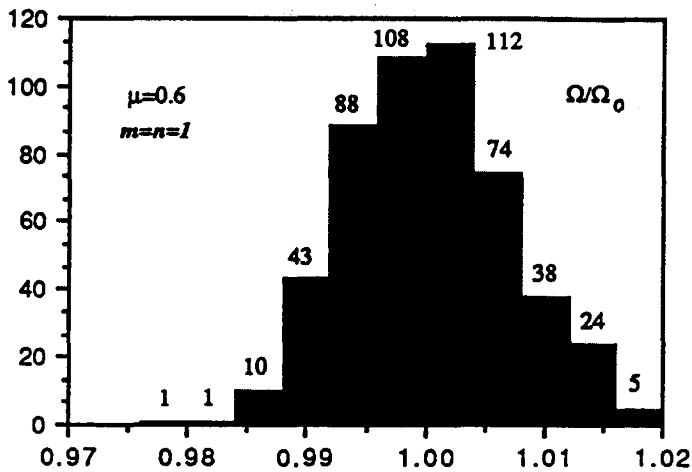

Fig. 5. Histogram of frequency ratio.

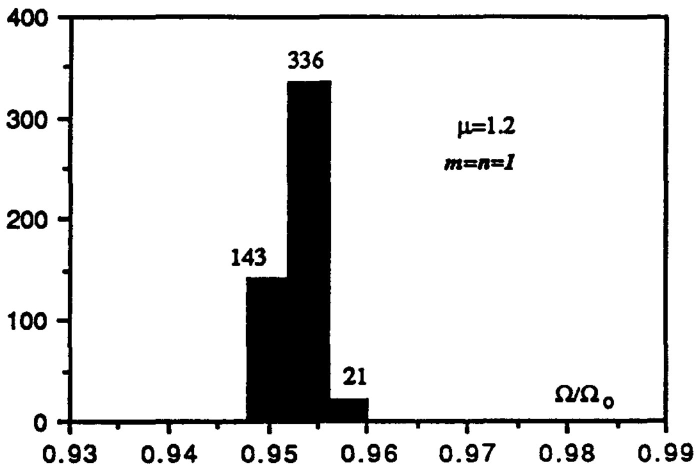

Fig. 6. Histogram of frequency ratio.

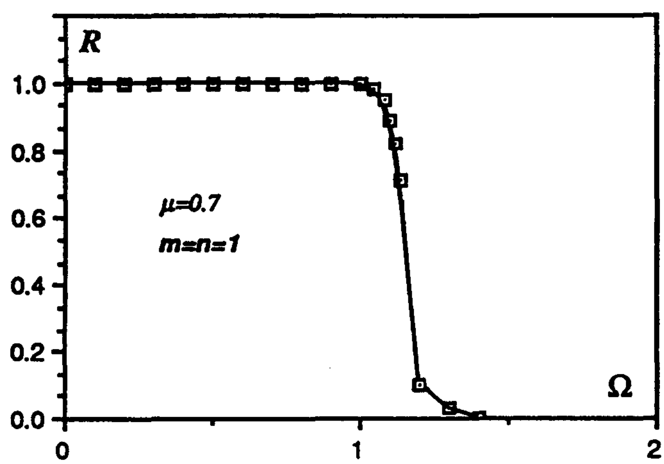

Fig. 7. The frequency reliability.

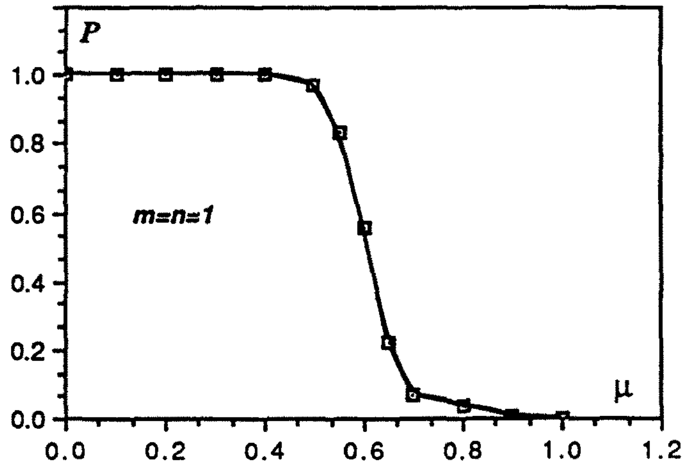

Fig. 8. The hardening type probability.

The *reliability function* is defined as the probability that the random natural frequency of a plate structure is greater than a specific frequency value. The reliability function measures physically the reliability that no consonance would occur when the structure is in the environment of the specific vibration frequency. A more general definition of the reliability function, i.e. the probability that the random natural frequency lies in some specific region, can also be considered. The reliability function of frequency distribution at $\mu = 0.7$ and $m = n = 1$ is given in Fig. 7. For example, the reliability of $\Omega > 1.14$ is 0.712, i.e. $\mathrm{Prob}(\Omega > 1.14) = 0.712$. The *hardening type probability* is defined to be the probability that the vibration behavior of a given plate is of hardening type. Figure 8 gives the hardening type probability as a function of the mean imperfection amplitude μ at $m = n = 1$. The probability curve indicates clearly that for small imperfection amplitudes ($\mu < 0.6$) most of the plates are of hardening type (since Prob [*plate is hardening type* | $\mu < 0.6$] > 0.5) and for large imperfection amplitudes ($\mu \geqslant 0.6$) most of the plates are of softening type (since Prob [*plate is hardening type* | $\mu \geqslant 0.6$] < 0.5), a result consistent with the works on nonlinear vibrations of the perfect plates ($\mu = 0$) in Chu and Herrmann (1956), Yamaki (1961) and Prathap and Varadan (1978), and the corresponding works for imperfect plates ($\mu = 0.5$) in Hui and Leissa (1983a, b) and Hui (1983, 1984a).

According to the *Kolmogorov–Smirnov* test (Massey, 1951) of goodness of fit at a level of significance of 0.05 the *critical value* of the maximum absolute difference between the unknown theoretical and obtained simulated distributions of Ω and $\Omega\,\Omega_0$ is $1.36/\sqrt{500} = 0.0608$.

6. CONCLUSION

A statistical method using the Monte Carlo technique for the analysis of the effect of random initial geometric imperfections on the vibrational behavior of rectangular plates is presented. The proposed method is adequate and practical to deal with the uncertainties of the realistic imperfections and can be used to solve the random geometric imperfection representation and the corresponding vibration analysis in a unified way. The *Monte Carlo* analysis performed for the simply supported plates illustrates how to use the proposed approach to calculate the frequency reliability function and the probability by which a plate behaves like a hard-spring or a soft-spring, the two characteristics which can be used to describe quantitatively the effect of random geometric imperfections on the vibration behavior of the plates.

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REFERENCES

Chia, C. Y. (1980). *Nonlinear Analysis of Plates*. McGraw-Hill, New York.

Chu, H. N. (1961). Influence of large amplitudes on flexural vibrations of a thin circular cylindrical shell. *J. Aerospace Sci.* **28**, 602–609.

Chu, H. N. and Herrmann, G. (1956). Influence of large amplitudes on free flexural vibrations of rectangular elastic plates. *ASME J. Appl. Mech.* **23**, 532–540.

Elishakoff, I. (1979). Simulation of space–random fields for solution of stochastic boundary-value problems. *J. Acoust. Soc. Am.* **60**(2), 399–403.

Elishakoff, I. and Arbocz, J. (1982). Reliability of axially compressed cylindrical shells with random axisymmetric imperfections. *Int. J. Solids Structures* **18**(7), 563–585.

Elishakoff, I. (1983). *Probabilistic Methods in the Theory of Structures*. John Wiley, New York.

Hui, D. (1983). Large amplitude axisymmetric vibrations of geometrically imperfect circular plates. *J. Sound Vib.* **91**(2), 239–246.

Hui, D. (1984a). Effects of geometric imperfections on large amplitude vibrations of rectangular flat plates with hysteresis damping. *ASME J. Appl. Mech.* **51**(1), 216–220.

Hui, D. (1984b). Influence of geometric imperfections and in-plane constraints on nonlinear vibrations of simply supported cylindrical panels. *ASME J. Appl. Mech.* **51**(2), 383–390.

Hui, D. and Leissa, A. W. (1983a). Effects of uni-directional geometric imperfections on vibrations of pressurized shallow spherical shells. *Int. J. Nonlinear Mech.* **18**(4), 279–286.

Hui, D. and Leissa, A. W. (1983b). Effects of geometric imperfections on vibrations of biaxially compressed rectangular flat plates. *ASME J. Appl. Mech.* **50**(4), 750–758.

Massey, F. J. Jr. (1951). The Kolmogorov–Smirnov test for goodness of fit. *J. Am. Statist. Assoc.* **46**, 68–78.

Mihram, G. A. (1972). *Simulation: Statistical Foundations and Methodology*. Academic Press, New York.

Nayfeh, A. H. and Mook, D. T. (1979). *Nonlinear Oscillations*. John Wiley, New York.

Prathap, G. and Varadan, T. K. (1978). On the non-linear vibrations of rectangular plates. *J. Sound Vib.* **56**(4), 521–530. Also, see Letter to the Editor, *J. Sound Vib.* **61**(2), 312–314.

Reissner, E. (1955). Non-linear effects in vibrations of cylindrical shells. The Woodridge Corporation, Guided Missile Research Division, Aeromechanics Section, Report AM 5-6, 63 pp.

Rosen, A. and Singer, J. (1974). Effect of axisymmetric imperfections on the vibration of cylindrical shells under axial compression. *AIAA Jl* **12**, 995–997.

Singer, J. and Prucz, J. (1982). Influence of initial geometric imperfections on vibrations of axially compressed stiffened cylindrical shells. *J. Sound Vib.* **80**(1), 117–143.

Wang, F. Y. (1986). Large amplitude vibrations of rectangular plates with random geometric imperfections. *Proc. International Conf. on Vibration Problems in Engineerings*, Xian, China, pp. 385–390.

Yamaki, N. (1961). Influence of large amplitudes on flexural vibrations of elastic plates. *Z. Angew. Math. Mech.* 501–510.